Statistics for Business and Economics

8th Global Edition

Chapter 15

Analysis of Variance

Chapter Goals

After completing this chapter, you should be able to:

- Recognize situations in which to use analysis of variance
- Understand different analysis of variance designs
- Perform a one-way and two-way analysis of variance and interpret the results
- Conduct and interpret a Kruskal-Wallis test
- Analyze two-factor analysis of variance tests with more than one observation per cell

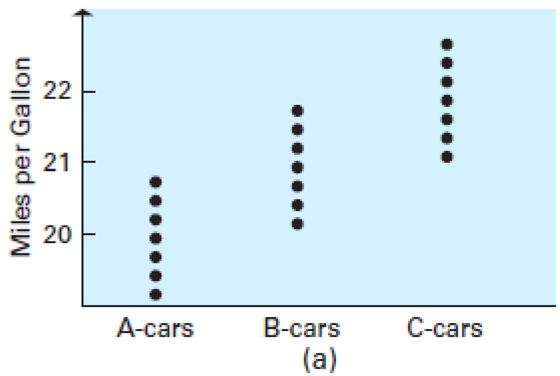
15.1

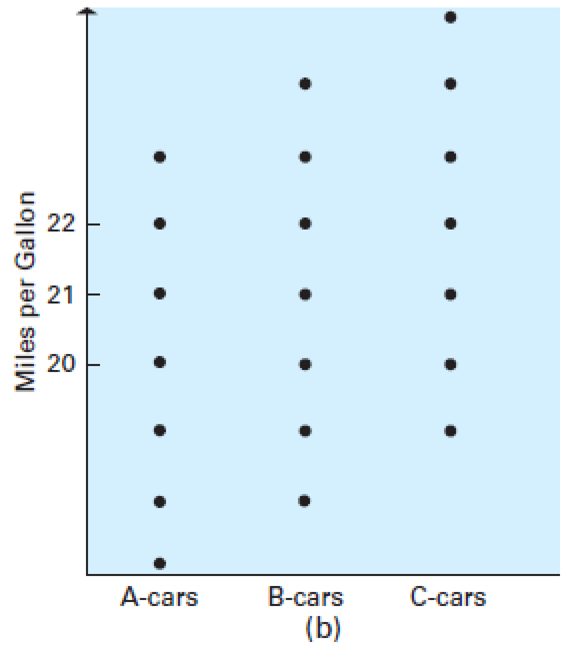
Comparison of Several Population Means

- Tests were presented in Chapter 10 for the difference between two population means
- In this chapter these procedures are extended to tests for the equality of more than two population means
- The null hypothesis is that the population means are all the same
- The critical factor is the variability involved in the data
  - If the variability around the sample means is small compared with the variability among the sample means, we reject the null hypothesis

Comparison of Several Population Means

(continued)





- Small variation around the sample means compared to the variation among the sample means

- Large variation around the sample means compared to the variation among the sample means

15.2

One-Way Analysis of Variance

- Evaluate the difference among the means of three or more groups

Examples:	Average production for 1st, 2nd, and 3rd shifts
	Expected mileage for five brands of tires

- Assumptions
  - Populations are normally distributed
  - Populations have equal variances
  - Samples are randomly and independently drawn

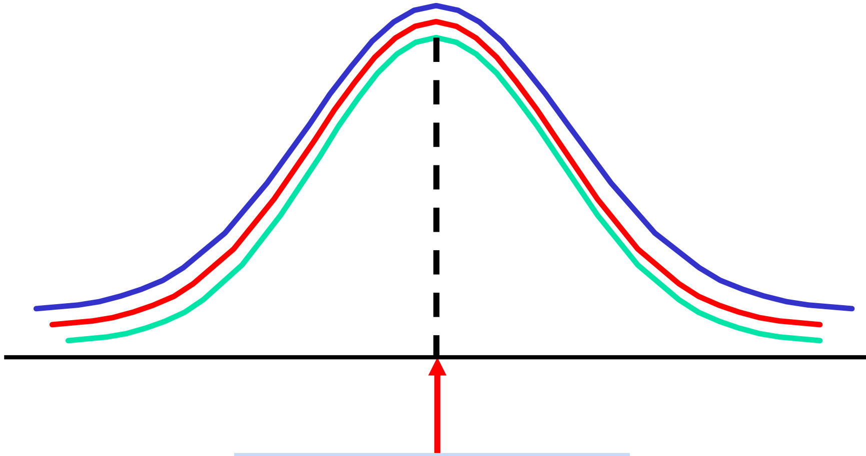
Hypotheses of One-Way ANOVA

- $H_0: \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_K$
  - All population means are equal
  - i.e., no variation in means between groups
- $H_1: \mu_i \neq \mu_j \quad \text{for at least one } i,j \text{ pair}$
  - At least one population mean is different
  - i.e., there is variation between groups
  - Does not mean that all population means are different (some pairs may be the same)

One-Way ANOVA

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_K$$

$$H_1 : \text{Not all } \mu_i \text{ are the same}$$



All Means are the same:
The Null Hypothesis is True
(No variation between groups)

$$\mu_1 = \mu_2 = \mu_3$$

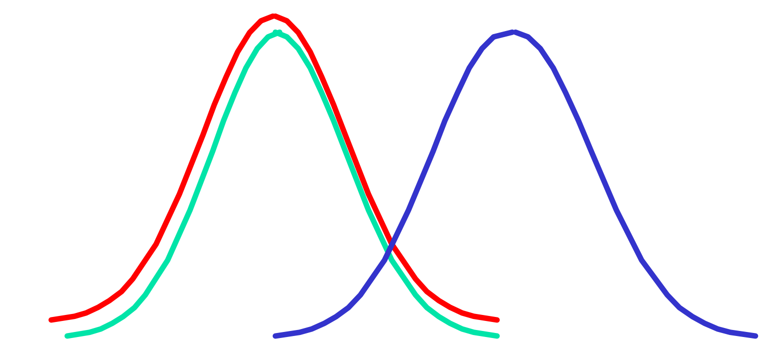
One-Way ANOVA

(continued)

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_K$$

$$H_1 : \text{Not all } \mu_i \text{ are the same}$$

At least one mean is different:
The Null Hypothesis is NOT true
(Variation is present between groups)



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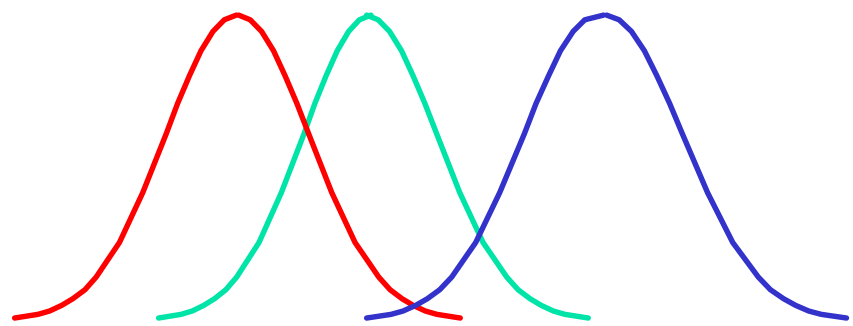
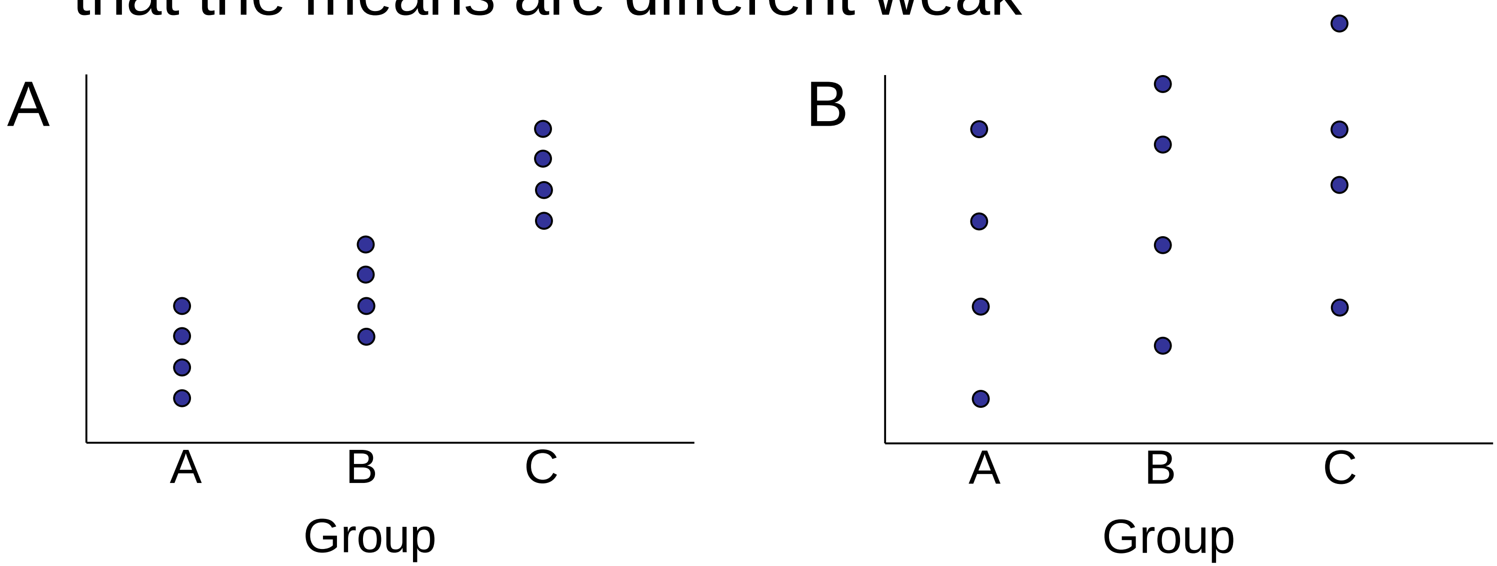


$\mu_1 = \mu_2 \neq \mu_3$

$\mu_1 \neq \mu_2 \neq \mu_3$

Variability

- The variability of the data is key factor to test the equality of means
- In each case below, the means may look different, but a large variation within groups in B makes the evidence that the means are different weak



Small variation within groups

Large variation within groups

Sum of Squares Decomposition

- Total variation can be split into two parts:

SST = SSW + SSG

SST = Total Sum of Squares

Total Variation = the aggregate dispersion of the individual data values across the various groups

SSW = Sum of Squares Within Groups

Within-Group Variation = dispersion that exists among the data values within a particular group

SSG = Sum of Squares Between Groups

Between-Group Variation = dispersion between the group sample means

Sum of Squares Decomposition

Total Sum of Squares (SST)

= **Variation due to random sampling (SSW)** + **Variation due to differences between groups (SSG)**

Total Sum of Squares

$$SST = SSW + SSG$$

$$SST = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (x_{ij} - \bar{\bar{x}})^2$$

Where:

SST = Total sum of squares

K = number of groups (levels or treatments)

n_i = number of observations in group i

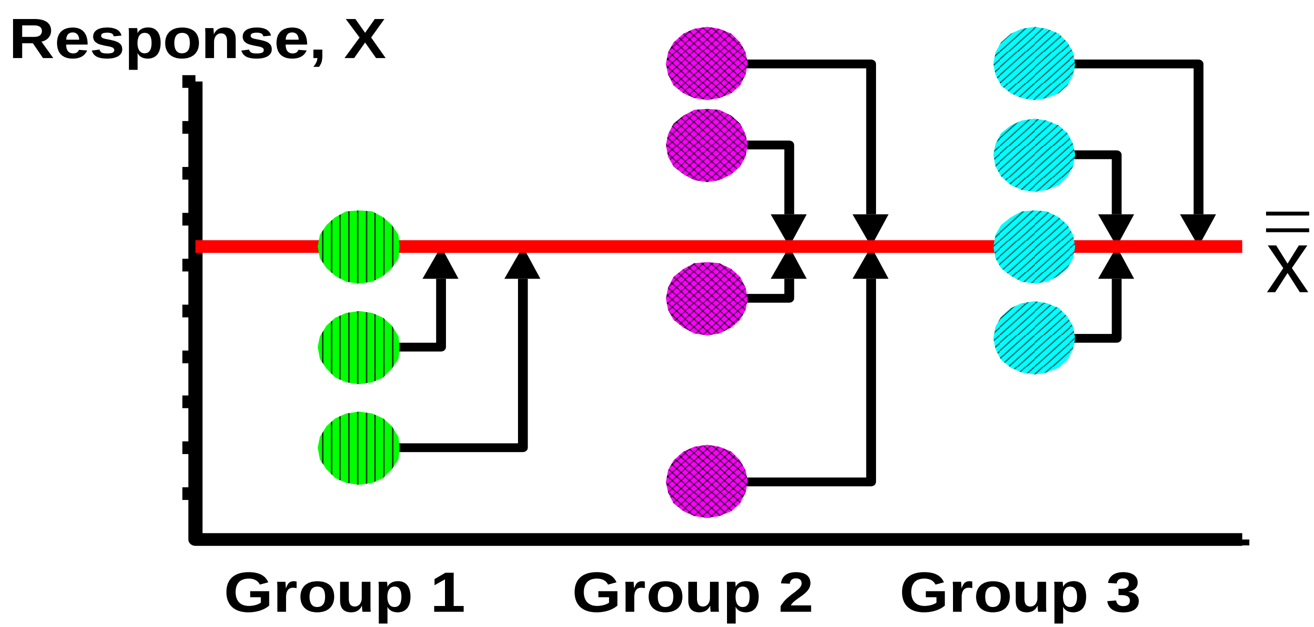
x_{ij} = j^{th} observation from group i

$\bar{\bar{x}}$ = overall sample mean

Total Sum of Squares

(continued)

$$SST = (X_{11} - \bar{\bar{X}})^2 + (X_{12} - \bar{\bar{X}})^2 + ... + (X_{Kn_K} - \bar{\bar{X}})^2$$



Within-Group Variation

$$SST = \boxed{SSW} + SSG$$

$$SSW = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2$$

Where:

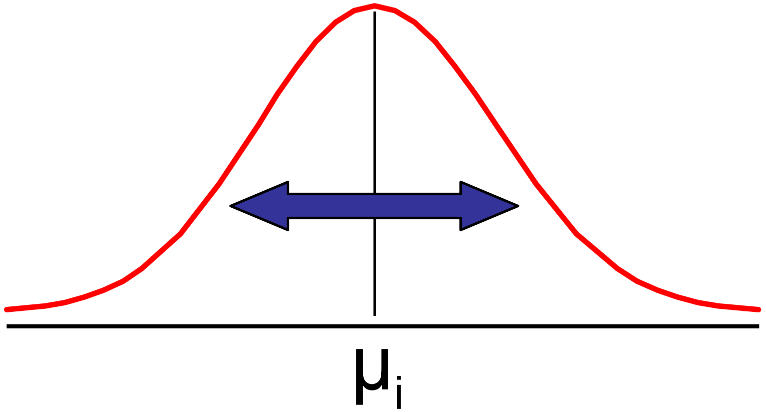
- SSW = Sum of squares within groups
- K = number of groups
- n_i = sample size from group i
- $\bar{x}_i$ = sample mean from group i
- x_{ij} = j^{th} observation in group i

Within-Group Variation

(continued)

$$SSW = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2$$

Summing the variation within each group and then adding over all groups



$$MSW = \frac{SSW}{n - K}$$

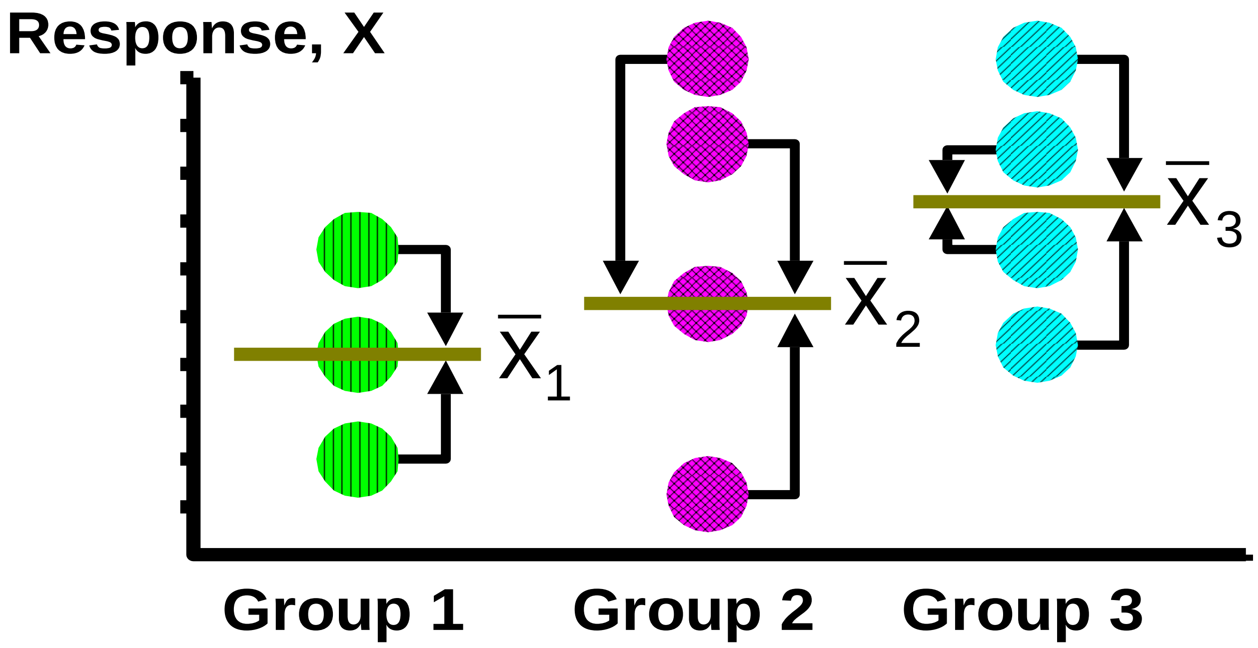
Mean Square Within =

SSW/degrees of freedom

Within-Group Variation

(continued)

$$SSW = (x_{11} - \bar{x}_1)^2 + (x_{12} - \bar{x}_1)^2 + \ldots + (x_{Kn_K} - \bar{x}_K)^2$$



Between-Group Variation

$$SST = SSW + SSG$$

$$SSG = \sum_{i=1}^{K} n_i(\bar{x}_i - \bar{\bar{x}})^2$$

Where:

SSG = Sum of squares between groups

K = number of groups

n_i = sample size from group i

$\bar{x}_i$ = sample mean from group i

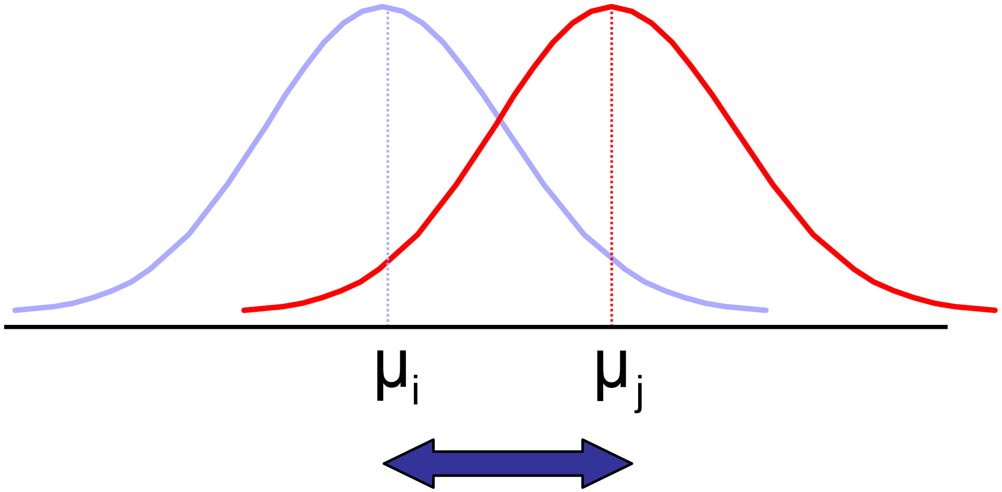
$\bar{\bar{x}}$ = grand mean (mean of all data values)

Between-Group Variation

(continued)

$$SSG = \sum_{i=1}^{K} n_i (\bar{x}_i - \bar{\bar{x}})^2$$

Variation Due to
Differences Between Groups



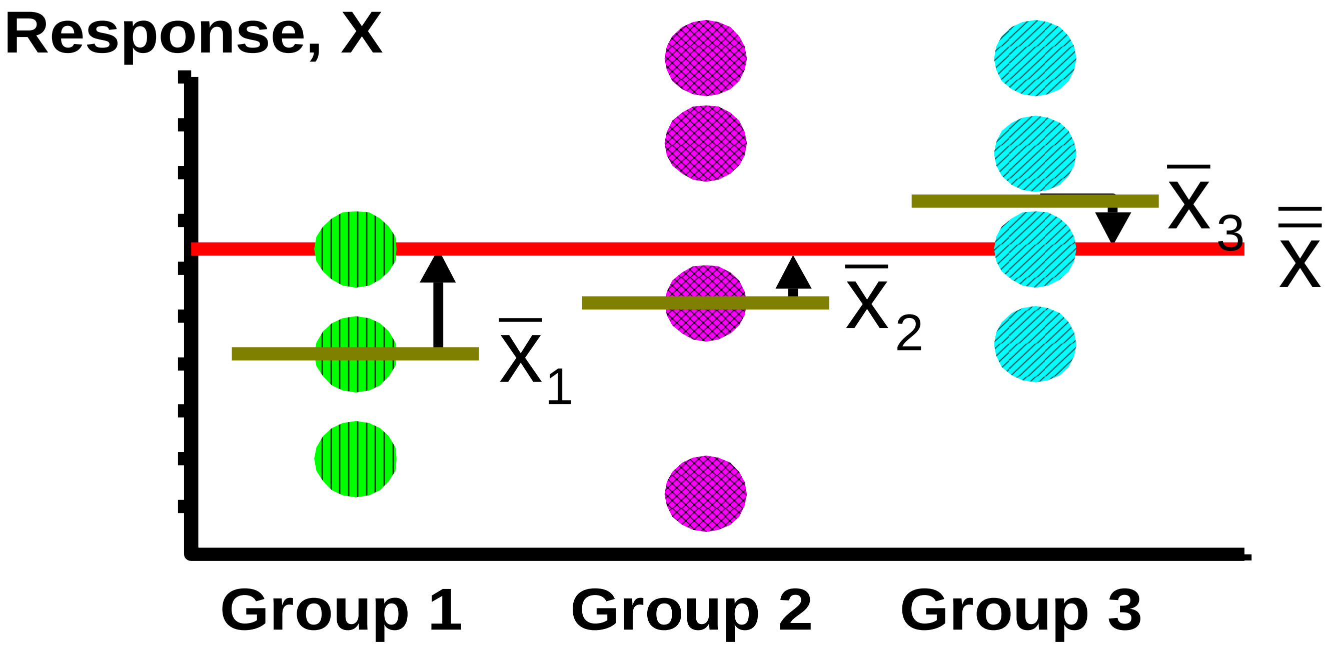
$$MSG = \frac{SSG}{K-1}$$

Mean Square Between Groups
= SSG/degrees of freedom

Between-Group Variation

(continued)

$$SSG = n_1(\bar{x}_1 - \bar{\bar{x}})^2 + n_2(\bar{x}_2 - \bar{\bar{x}})^2 + \ldots + n_K(\bar{x}_K - \bar{\bar{x}})^2$$



Obtaining the Mean Squares

$$MST = \frac{SST}{n-1}$$

$$MSW = \frac{SSW}{n-K}$$

$$MSG = \frac{SSG}{K-1}$$

Where n = sum of the sample sizes from all groups
K = number of populations

One-Way ANOVA Table

Source of Variation	SS	df	MS (Variance)	F ratio
Between Groups	SSG	K - 1	$MSG = \frac{SSG}{K - 1}$	$F = \frac{MSG}{MSW}$
Within Groups	SSW	n - K	$MSW = \frac{SSW}{n - K}$	
Total	SST = SSG+SSW	n - 1		

K = number of groups

n = sum of the sample sizes from all groups

df = degrees of freedom

One-Factor ANOVA F Test Statistic

$H_0: \mu_1 = \mu_2 = \ldots = \mu_K$

H_1: At least two population means are different

- Test statistic

$$F = \frac{MSG}{MSW}$$

MSG is mean squares between variances

MSW is mean squares within variances

- Degrees of freedom
  - $df_1 = K - 1$ (K = number of groups)
  - $df_2 = n - K$ (n = sum of sample sizes from all groups)

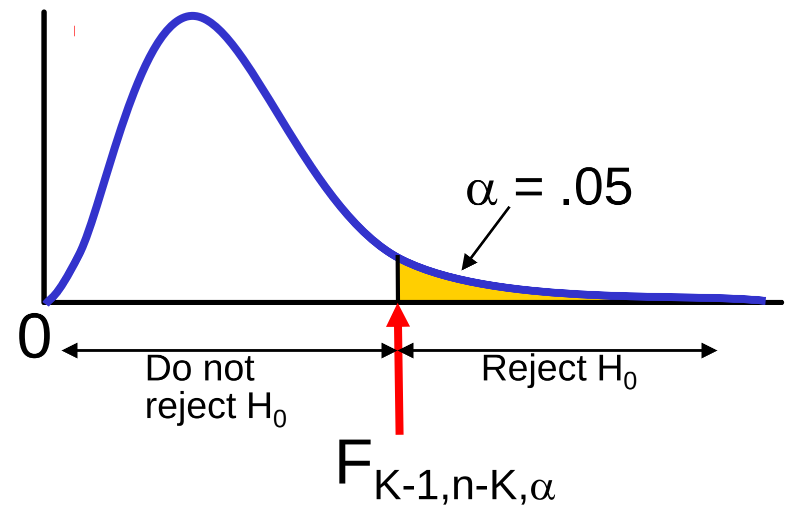
Interpreting the F Statistic

- The F statistic is the ratio of the between estimate of variance and the within estimate of variance
  - The ratio must always be positive
  - $df_1 = K - 1$ will typically be small
  - $df_2 = n - K$ will typically be large

Decision Rule:

- Reject H_0 if

  $F > F_{K-1,n-K,\alpha}$



One-Factor ANOVA F Test Example

You want to see if three different golf clubs yield different distances. You randomly select five measurements from trials on an automated driving machine for each club. At the .05 significance level, is there a difference in mean distance?

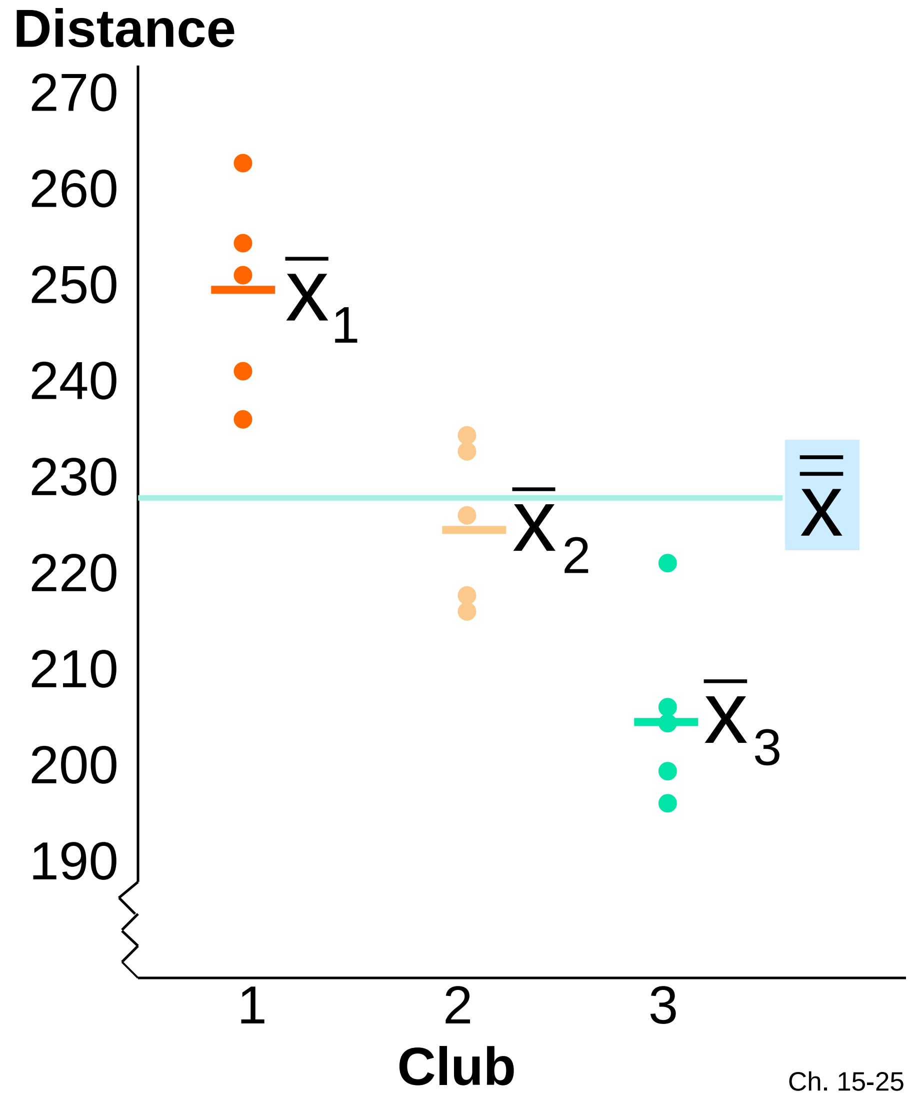
Club 1	**Club 2**	**Club 3**
254	234	200
263	218	222
241	235	197
237	227	206
251	216	204

One-Factor ANOVA Example: Scatter Diagram

Club 1	Club 2	Club 3
254	234	200
263	218	222
241	235	197
237	227	206
251	216	204

$\bar{x}_1 = 249.2$ | $\bar{x}_2 = 226.0$ | $\bar{x}_3 = 205.8$

$\bar{\bar{x}} = 227.0$



One-Factor ANOVA Example Computations

Club 1	Club 2	Club 3
254	234	200
263	218	222
241	235	197
237	227	206
251	216	204

$\bar{x}_1 = 249.2$ $n_1 = 5$

$\bar{x}_2 = 226.0$ $n_2 = 5$

$\bar{x}_3 = 205.8$ $n_3 = 5$

$n = 15$

$\bar{\bar{x}} = 227.0$

$K = 3$

$SSG = 5\,(249.2 - 227)^2 + 5\,(226 - 227)^2 + 5\,(205.8 - 227)^2 = 4716.4$

$SSW = (254 - 249.2)^2 + (263 - 249.2)^2 + \ldots + (204 - 205.8)^2 = 1119.6$

$MSG = 4716.4 / (3-1) = 2358.2$

$MSW = 1119.6 / (15-3) = 93.3$

$$F = \frac{2358.2}{93.3} = 25.275$$

One-Factor ANOVA Example Solution

$H_0: \mu_1 = \mu_2 = \mu_3$

$H_1: \mu_i$ not all equal

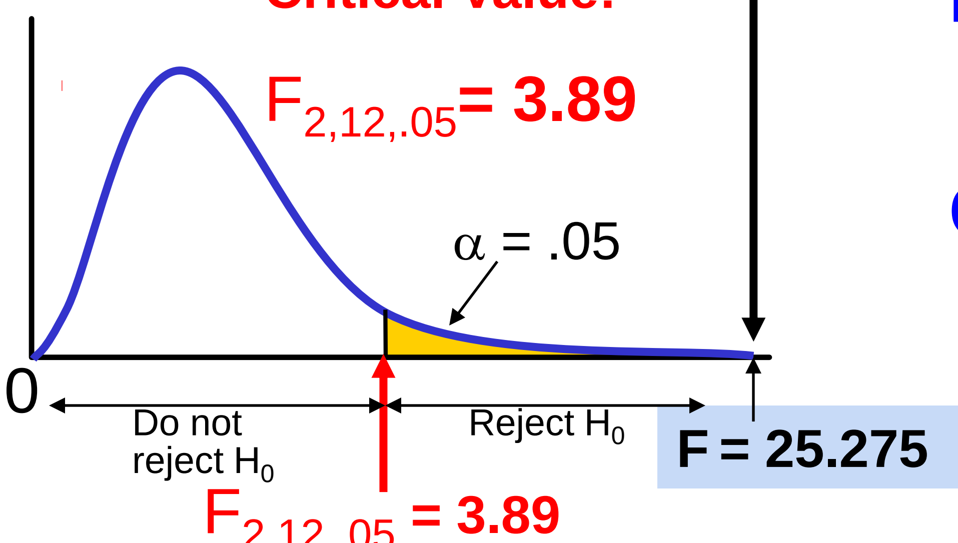
$\alpha = .05$

$df_1 = 2 \qquad df_2 = 12$

Test Statistic:

$$F = \frac{MSA}{MSW} = \frac{2358.2}{93.3} = 25.275$$

Critical Value:



Decision:

Reject H_0 at $\alpha = 0.05$

Conclusion:

There is evidence that at least one μ_i differs from the rest

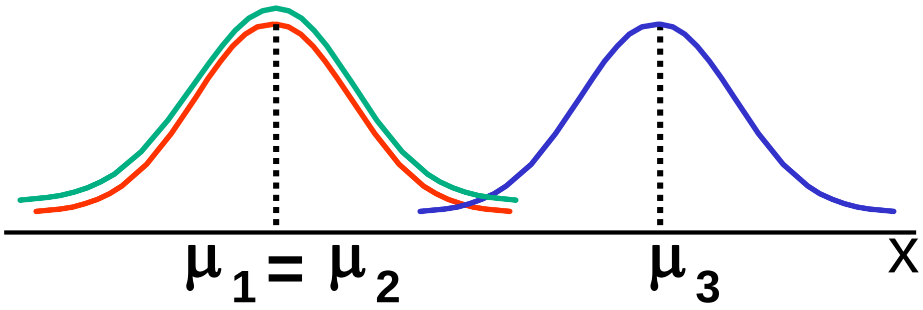
ANOVA -- Single Factor: Excel Output

EXCEL: data | data analysis | ANOVA: single factor

SUMMARY						
Groups	*Count*	*Sum*	*Average*	*Variance*		
Club 1	5	1246	249.2	108.2		
Club 2	5	1130	226	77.5		
Club 3	5	1029	205.8	94.2		
ANOVA						
Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	**4716.4**	**2**	**2358.2**	**25.275**	4.99E-05	**3.89**
Within Groups	**1119.6**	**12**	**93.3**			
Total	5836.0	14				

Multiple Comparisons Between Subgroup Means

- To test which population means are significantly different
  - e.g.: $\mu_1 = \mu_2 \neq \mu_3$
  - Done after rejection of equal means in single factor ANOVA design
- Allows pair-wise comparisons
  - Compare absolute mean differences with critical range



Two Subgroups

- When there are only two subgroups, compute the minimum significant difference (MSD)

$$MSD = t_{\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

Where s_p is a pooled estimate of the variance

- Use hypothesis testing methods of Ch. 10

Multiple Supgroups

- The minimum significant difference between k subgroups is

$$\mathrm{MSD}(K) = Q\frac{s_p}{\sqrt{n}}$$ where $$s_p = \sqrt{\mathrm{MSW}}$$

  - Q is a factor from Appendix Table 13 for the chosen level of α
  - K = number of subgroups, and
  - MSW = Mean square within from ANOVA table

Multiple Supgroups

(continued)

$$MSD(K) = Q\frac{s_p}{\sqrt{n}}$$

Compare:

$|\bar{x}_1 - \bar{x}_2|$

$|\bar{x}_1 - \bar{x}_3|$

$|\bar{x}_2 - \bar{x}_3|$

etc...

$$\text{Is } |\bar{x}_i - \bar{x}_j| > MSD(K) \text{ ?}$$

If the absolute mean difference is greater than MSD then there is a significant difference between that pair of means at the chosen level of significance.

Multiple Supgroups: Example

$\bar{x}_1 = 249.2$	$n_1 = 5$
$\bar{x}_2 = 226.0$	$n_2 = 5$
$\bar{x}_3 = 205.8$	$n_3 = 5$

$$MSD(K) = Q\frac{s_p}{\sqrt{n}} = 3.77\frac{\sqrt{93.3}}{\sqrt{15}} = 9.387$$

(where Q = 3.77 is from Table 13 for α = .05 and 12 df)

$$|\bar{x}_1 - \bar{x}_2| = 23.2$$

$$|\bar{x}_1 - \bar{x}_3| = 43.4$$

$$|\bar{x}_2 - \bar{x}_3| = 20.2$$

Since each difference is greater than 9.387, we conclude that all three means are different from one another at the .05 level of significance.

15.3

Kruskal-Wallis Test

- Use when the normality assumption for one-way ANOVA is violated
- Assumptions:
  - The samples are random and independent
  - variables have a continuous distribution
  - the data can be ranked
  - populations have the same variability
  - populations have the same shape

Kruskal-Wallis Test Procedure

- Obtain relative rankings for each value
  - In event of tie, each of the tied values gets the average rank
- Sum the rankings for data from each of the K groups
  - Compute the Kruskal-Wallis test statistic
  - Evaluate using the chi-square distribution with K – 1 degrees of freedom

Kruskal-Wallis Test Procedure

(continued)

- The Kruskal-Wallis test statistic:

  (chi-square with K – 1 degrees of freedom)

$$W = \left[\frac{12}{n(n+1)} \sum_{i=1}^{K} \frac{R_i^2}{n_i} \right] - 3(n+1)$$

where:

n = sum of sample sizes in all groups

K = Number of samples

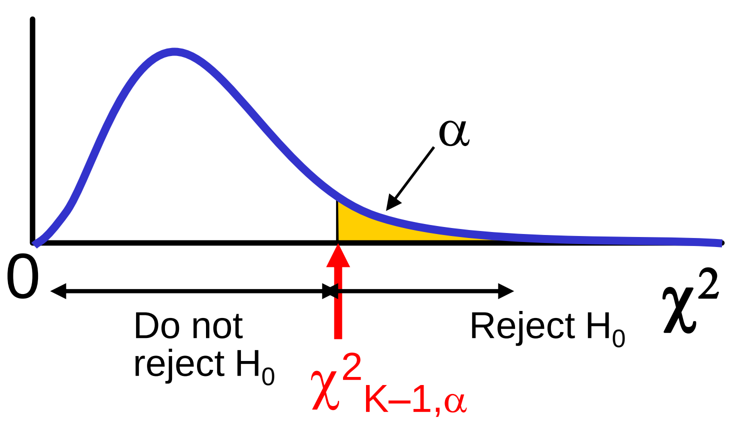
R_i = Sum of ranks in the i^{th} group

n_i = Size of the i^{th} group

Kruskal-Wallis Test Procedure

(continued)

- Complete the test by comparing the calculated H value to a critical χ^2 value from the chi-square distribution with K – 1 degrees of freedom



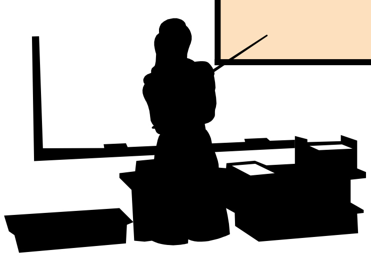
Decision rule

- Reject H_0 if $W > \chi^2_{K-1,\alpha}$
- Otherwise do not reject H_0

Kruskal-Wallis Example

- Do different departments have different class sizes?

Class size (Math, M)	Class size (English, E)	Class size (Biology, B)
23	55	30
41	60	40
54	72	18
78	45	34
66	70	44

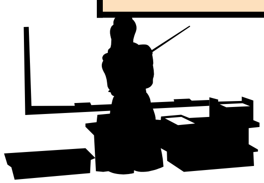


Size	Rank
18	1
23	2
30	3
34	4
40	5
41	6
44	7
45	8
54	9
55	10
60	11
66	12
70	13
72	14
78	15

Kruskal-Wallis Example

- Do different departments have different class sizes?

Class size (Math, M)	Ranking	Class size (English, E)	Ranking	Class size (Biology, B)	Ranking
23	2	55	10	30	3
41	6	60	11	40	5
54	9	72	14	18	1
78	15	45	8	34	4
66	12	70	13	44	7
	$\Sigma = 44$		$\Sigma = 56$		$\Sigma = 20$



Kruskal-Wallis Example

(continued)

$$H_0: \text{Mean}_M = \text{Mean}_E = \text{Mean}_B$$

$$H_1: \text{Not all population means are equal}$$

- The W statistic is

$$W = \left[\frac{12}{n(n+1)} \sum_{i=1}^{K} \frac{R_i^2}{n_i} \right] - 3(n+1)$$

$$= \left[\frac{12}{15(15+1)} \left(\frac{44^2}{5} + \frac{56^2}{5} + \frac{20^2}{5} \right) \right] - 3(15+1) = 6.72$$

Kruskal-Wallis Example

(continued)

- Compare $W = 6.72$ to the critical value from the chi-square distribution for 3 – 1 = 2 degrees of freedom and α = .05:

$$\chi^2_{2,0.05} = 5.991$$

Since $H = 6.72$ **>** $\chi^2_{2,0.05} = 5.991$,

reject H_0

There is sufficient evidence to reject that the population means are all equal

15.4

Two-Way Analysis of Variance

One Observation per Cell, Randomized Blocks

- Examines the effect of
  - Two factors of interest on the dependent variable
    - e.g., Percent carbonation and line speed on soft drink bottling process
  - Interaction between the different levels of these two factors
    - e.g., Does the effect of one particular carbonation level depend on which level the line speed is set?

Two-Way ANOVA

(continued)

- Assumptions
  - Populations are normally distributed
  - Populations have equal variances
  - Independent random samples are drawn

Randomized Block Design

Two Factors of interest: A and B

K = number of groups of factor A

H = number of levels of factor B

(sometimes called a blocking variable)

Block	Group			
	1	2	…	K
1	x_{11}	x_{21}	…	x_{K1}
2	x_{12}	x_{22}	…	x_{K2}
.	.	.	.	.
.	.	.	.	.
H	x_{1H}	x_{2H}	…	x_{KH}

Two-Way Notation

- Let x_{ij} denote the observation in the i^{th} group and j^{th} block
- Suppose that there are K groups and H blocks, for a total of n = KH observations
- Let the overall mean be $\bar{\bar{x}}$
- Denote the group sample means by

$$\bar{x}_{i\bullet} \quad (i = 1, 2, \ldots, K)$$

- Denote the block sample means by

$$\bar{x}_{\bullet j} \quad (j = 1, 2, \ldots, H)$$

Partition of Total Variation

- SST = SSG + SSB + SSE

Total Sum of Squares (SST)	=	Variation due to differences between groups (SSG)
		+
		Variation due to differences between blocks (SSB)
		+
The error terms are assumed to be independent, normally distributed, and have the same variance	→	Variation due to random sampling (unexplained error) (SSE)

Two-Way Sums of Squares

- The sums of squares are

Degrees of Freedom:

Total: $$SST = \sum_{i=1}^{K}\sum_{j=1}^{H}(x_{ij} - \bar{\bar{x}})^2$$ $n - 1$

Between-Groups: $$SSG = H\sum_{i=1}^{K}(\bar{x}_{i\bullet} - \bar{\bar{x}})^2$$ $K - 1$

Between-Blocks: $$SSB = K\sum_{j=1}^{H}(\bar{x}_{\bullet j} - \bar{\bar{x}})^2$$ $H - 1$

Error: $$SSE = \sum_{i=1}^{K}\sum_{j=1}^{H}(x_{ij} - \bar{x}_{i\bullet} - \bar{x}_{\bullet j} + \bar{\bar{x}})^2$$ $(K - 1)(H - 1)$

Two-Way Mean Squares

- The mean squares are

$$MST = \frac{SST}{n-1}$$

$$MSG = \frac{SST}{K-1}$$

$$MSB = \frac{SST}{H-1}$$

$$MSE = \frac{SSE}{(K-1)(H-1)}$$

Two-Way ANOVA: The F Test Statistic

F Test for Groups

H_0: The K population group means are all the same

$$F = \frac{MSG}{MSE}$$

Reject H_0 if

$F > F_{K-1,(K-1)(H-1),\alpha}$

F Test for Blocks

H_0: The H population block means are the same

$$F = \frac{MSB}{MSE}$$

Reject H_0 if

$F > F_{H-1,(K-1)(H-1),\alpha}$

General Two-Way Table Format

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	F Ratio
Between groups	SSG	K – 1	$MSG = \frac{SSG}{K-1}$	$\frac{MSG}{MSE}$
Between blocks	SSB	H – 1	$MSB = \frac{SSB}{H-1}$	$\frac{MSB}{MSE}$
Error	SSE	(K – 1)(H – 1)	$MSE = \frac{SSE}{(K-1)(H-1)}$	
Total	SST	n - 1		

More than One Observation per Cell

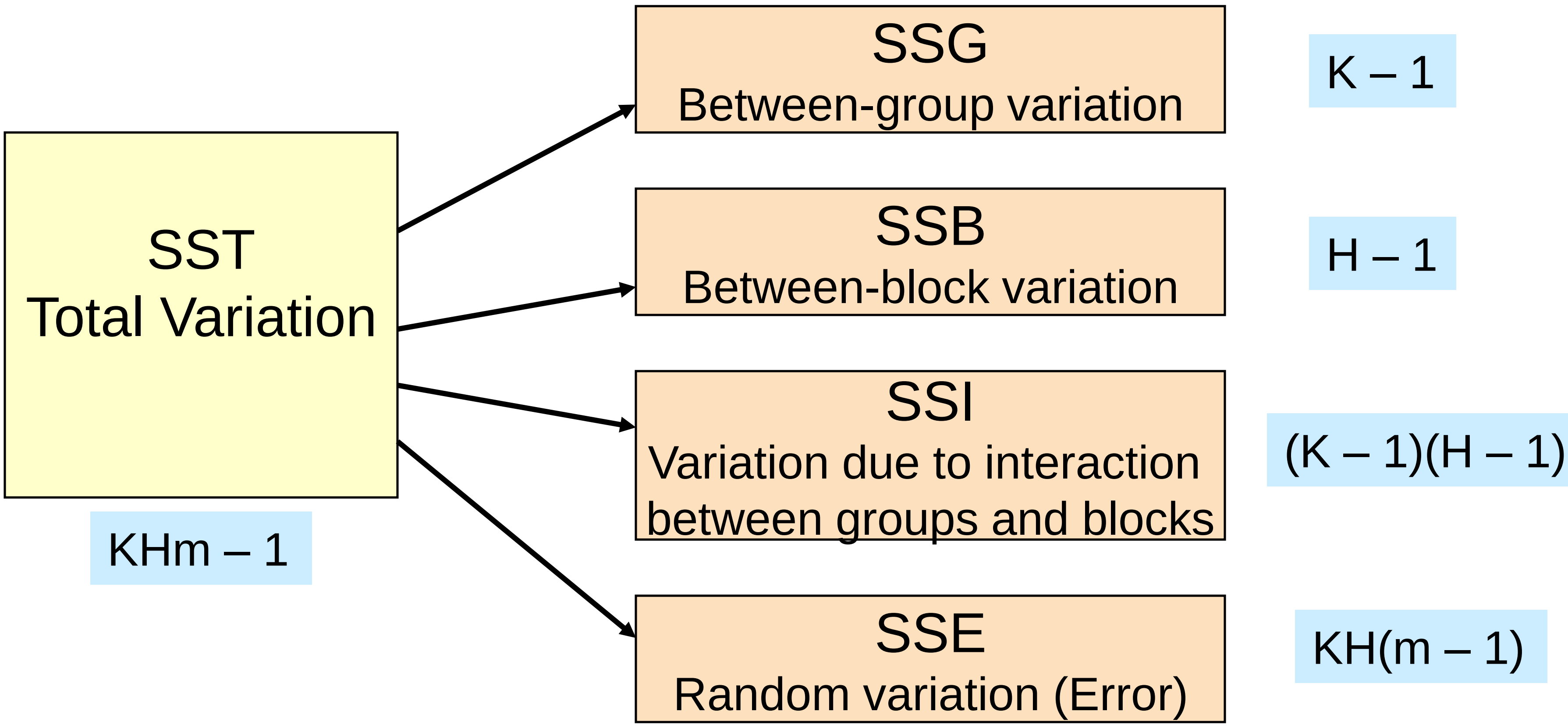
- A two-way design with more than one observation per cell allows one further source of variation
- The interaction between groups and blocks can also be identified
- Let
  - K = number of groups
  - H = number of blocks
  - m = number of observations per cell
  - KHm = total number of observations

More than One Observation per Cell

(continued)

SST = SSG + SSB + SSI + SSE

Degrees of Freedom:



Sums of Squares with Interaction

Degrees of Freedom:

Total: $SST = \sum_i \sum_j \sum_l (x_{ijl} - \bar{\bar{x}})^2$ — $KHm - 1$

Between-groups: $SSG = Hm \sum_{i=1}^{K} (\bar{x}_{i\bullet\bullet} - \bar{\bar{x}})^2$ — $K - 1$

Between-blocks: $SSB = Km \sum_{j=1}^{H} (x_{\bullet j\bullet} - \bar{\bar{x}})^2$ — $H - 1$

Interaction: $SSI = m \sum_{i=1}^{K} \sum_{j=1}^{H} (\bar{x}_{ij\bullet} - \bar{x}_{i\bullet\bullet} - \bar{x}_{\bullet j\bullet} + \bar{\bar{x}})^2$ — $(K - 1)(H - 1)$

Error: $SSE = \sum_i \sum_j \sum_l (x_{ijl} - \bar{x}_{ij\bullet})^2$ — $KH(m - 1)$

Two-Way Mean Squares with Interaction

- The mean squares are

$$MST = \frac{SST}{KHm-1}$$

$$MSG = \frac{SST}{K-1}$$

$$MSB = \frac{SST}{H-1}$$

$$MSI = \frac{SSI}{(K-1)(H-1)}$$

$$MSE = \frac{SSE}{KH(m-1)}$$

Two-Way ANOVA: The F Test Statistic

H_0: The K population group means are all the same

F Test for group effect

$$F = \frac{MSG}{MSE}$$

Reject H_0 if

$F > F_{K-1,KH(L-1),\alpha}$

H_0: The H population block means are the same

F Test for block effect

$$F = \frac{MSB}{MSE}$$

Reject H_0 if

$F > F_{H-1,KH(L-1),\alpha}$

H_0: the interaction of groups and blocks is equal to zero

F Test for interaction effect

$$F = \frac{MSI}{MSE}$$

Reject H_0 if

$F > F_{(K-1)(H-1),KH(L-1),\alpha}$

Two-Way ANOVA Summary Table

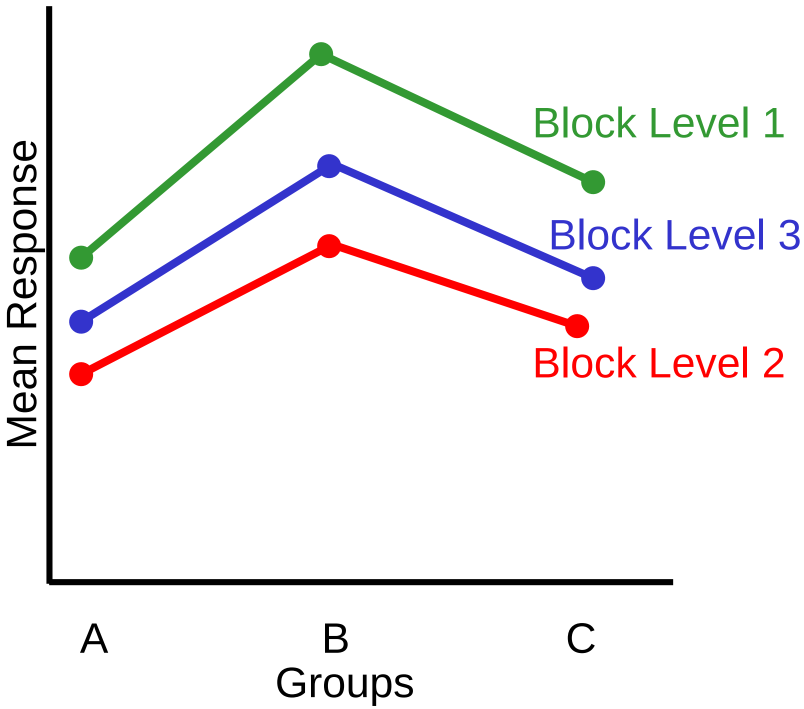
Source of Variation	Sum of Squares	Degrees of Freedom	Mean Squares	F Statistic
Between groups	SSG	K – 1	MSG = SSG / (K – 1)	$\frac{MSG}{MSE}$
Between blocks	SSB	H – 1	MSB = SSB / (H – 1)	$\frac{MSB}{MSE}$
Interaction	SSI	(K – 1)(H – 1)	MSI = SSI / (K – 1)(H – 1)	$\frac{MSI}{MSE}$
Error	SSE	KH(m – 1)	MSE = SSE / KH(m – 1)	
Total	SST	KHm – 1		

Features of Two-Way ANOVA F Test

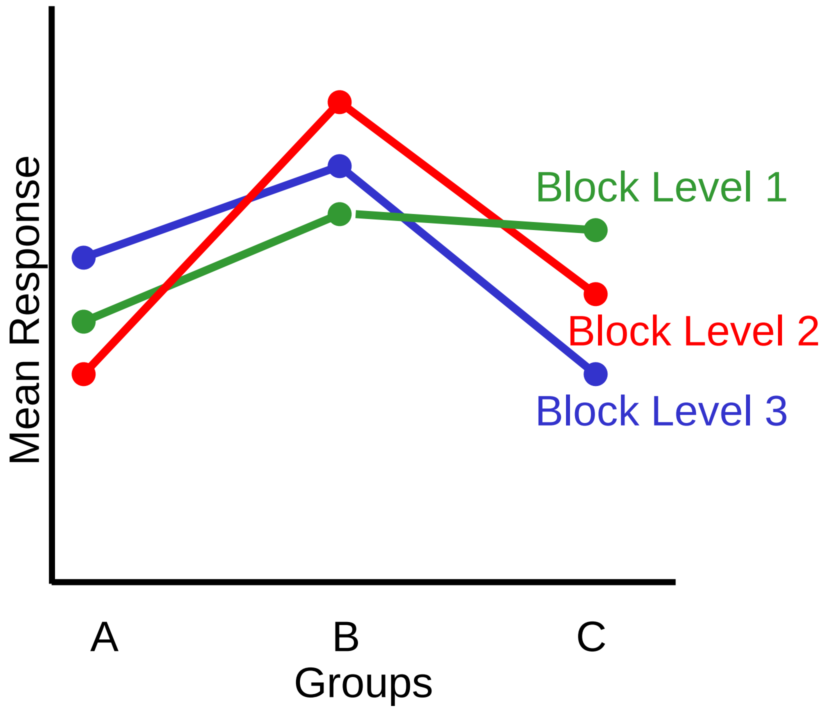
- Degrees of freedom always add up
  - KHm - 1 = (K-1) + (H-1) + (K-1)(H-1) + KH(m-1)
  - Total = groups + blocks + interaction + error
- The denominator of the F Test is always the same but the numerator is different
- The sums of squares always add up
  - SST = SSG + SSB + SSI + SSE
  - Total = groups + blocks + interaction + error

Examples:
Interaction vs. No Interaction

- No interaction:



- Interaction is present:



Chapter Summary

- Described one-way analysis of variance
  - The logic of Analysis of Variance
  - Analysis of Variance assumptions
  - F test for difference in K means
- Applied the Kruskal-Wallis test when the populations are not known to be normal
- Described two-way analysis of variance
  - Examined effects of multiple factors
  - Examined interaction between factors